C4 Exercise 6F (integration by substitution)

Note Title 01/10/2007

example 11

Use the substitution $u = 2x+5$ to find $\int x\sqrt{2x+5}\,dx$

Let $I = \int$ [illegible]

then put $u = 2x+5 \Rightarrow$ [illegible]

so [illegible] $\Rightarrow$ [illegible]

[illegible]

$$\text{So} \quad I = \int \left(\frac{u-5}{2}\right)\left(u^{1/2}\right)\left(\tfrac{1}{2}\,du\right) = \int \frac{1}{4}u^{3/2} - \frac{5}{4}u^{1/2}\,du$$

C4 Ex 6F

example 12

Use the substitution $u = \sin x + 1$ to find

$\int$ [illegible] ([illegible]

put $u = \sin x + 1 \Rightarrow \frac{du}{dx} = \cos x$

$\Rightarrow \sin x = u - 1$

So we can replace [illegible]

$$\int \cancel{\cos x}\,(u-1)\; u^3 \; \frac{du}{\cancel{\cos x}}$$

$$I = \int (u-1)\,u^3 \; du = \int u^4 - u^3 \; du = \frac{1}{5}u^5 - \frac{1}{4}u^4 + C$$

$$I = \frac{1}{5}(1+\sin x)^5 - \frac{1}{4}(1+\sin x)^4 + C$$

Exercise 6F.

1a $\int x\sqrt{1+x}\ dx$ using $u = 1+x$

$\Rightarrow x = u-1$ and $\frac{du}{dx} = 1$

so 'dx' can be replaced directly by 'du'

$I = \int (u-1)(u)^{1/2}\ du = \int \left(u^{3/2} - u^{1/2}\right) du$

$\Rightarrow I = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$\Rightarrow I = \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C$

1b $I = \int \frac{x}{\sqrt{1+x}} \, dx$ using $u = 1 + x$

$u = 1 + x \Rightarrow$ 'dx' can be replaced by 'du'

and $(u-1) = x$

so $\int \frac{x}{\sqrt{1+x}} \, dx = \int \frac{u-1}{u^{1/2}} \, du$

$= \int u^{1/2} - u^{-1/2} \, du$

$= \frac{2}{3} u^{3/2} - 2u^{1/2} + C$

$= \frac{2}{3}(1+x)^{3/2} - 2\sqrt{1+x} + C$

1c $\quad I = \int \frac{1+\sin x}{\cos x}\, dx \qquad$ using $u = \sin x$

$$u = \sin x \quad \Rightarrow \quad \frac{du}{dx} = \cos x$$

and so 'dx' can be replaced by '$\frac{1}{\cos x} du$'

$$\Rightarrow \quad I = \int \left(\frac{1+u}{\cos x}\right) \cdot \left(\frac{1}{\cos x}\right) du$$

$$I = \int \frac{(1+u)}{\cos^2 x}\, du$$

$$I = \int \frac{1+u}{1-\sin^2 x}\, du$$

1c (ctd)

$$\Rightarrow I = \int \frac{1+u}{1-u^2}\,du$$

$$\Rightarrow I = \int \frac{\cancel{1+u}\ 1}{\cancel{(1+u)}(1-u)}\,du$$

$$\Rightarrow I = \int \frac{1}{1-u}\,du$$

$$\int \frac{1}{ax+b}\,dx = a\ln|ax+b| + C$$

So $I = -\ln|1-u| + C$

$$I = -\ln|1-\sin x| + C$$

1d find $I = \int x(3+2x)^5\,dx$ using $u = 3+2x$

$u = 3+2x \Rightarrow \frac{du}{dx} = 2$ and hence 'dx' replaced by '$\frac{1}{2}du$'

$x = \frac{u-3}{2}$

$$\Rightarrow I = \int \left(\frac{u-3}{2}\right)(u)^5\left(\frac{1}{2}\right)du = \frac{1}{4}\int u^6 - 3u^5\,du$$

$$= \frac{1}{4}\left(\frac{1}{7}u^7 - \frac{1}{2}u^6\right) + C$$

$$= \frac{1}{28}(3+2x)^7 - \frac{1}{8}(3+2x)^6 + C$$

1e find $I = \int \sin^3 x \, dx$ using the substitution $u = \cos x$

$u = \cos x \Rightarrow \frac{du}{dx} = \sin x \Rightarrow$ 'dx' replaced by '$\frac{1}{\sin x} du$'

$$\Rightarrow I = \int \sin^3 x \left(\frac{1}{\sin x}\right) du$$

$$= \int \sin^2 x \, du$$

$$= \int 1 - \cos^2 x \, du$$

$$= \int 1 - u^2 \, du$$

1e (ctd)

$$I = u - \frac{1}{3}u^3 + C$$

$$I = \cos x - \frac{1}{3}\cos^3 x + C$$

2a $I = \int x\sqrt{2+x}\, dx$ using $u^2 = 2+x$

$\Rightarrow 2u\frac{du}{dx} = 1$

$\Rightarrow$ 'dx' replaced by '$2u\,du$'

and $x = u^2 - 2$

So $I = \int (u^2-2)(u^2)^{1/2}(2u)\,du$

$I = \int 2u^4 - 4u^2 \quad du$

$I = \frac{2}{5}u^5 - \frac{4}{3}u^3 + C$

$I = \frac{2}{5}(2+x)^{5/2} - \frac{4}{3}(2+x)^{3/2} + C$

2b find $I = \int \frac{2}{\sqrt{x}(x-4)} \, dx$ given substitution $u = \sqrt{x}$

$\Rightarrow \frac{du}{dx} = \frac{1}{2} x^{-1/2} \Rightarrow \frac{1}{\frac{1}{2}x^{-1/2}} \, du = dx$

$\Rightarrow$ 'dx' replaced by $2\sqrt{x} \, du$ or $2u \, du$

$u = \sqrt{x} \Rightarrow x = u^2$ and $x - 4 = u^2 - 4$

$$\Rightarrow I = \int \left(\frac{2}{\cancel{u}(u^2-4)} \right) (2\cancel{u}) \, du = \int \frac{4}{u^2-4} \, du$$

$$I = \int \frac{4}{u^2-4} \, du = \int \frac{4}{(u+2)(u-2)} \, du$$

now use partial fractions to rewrite $\frac{4}{(u+2)(u-2)}$

$$\frac{4}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$$

$$\Rightarrow \quad 4 = A(u-2) + B(u+2)$$

$u=2 \Rightarrow \quad 4 = 4B \Rightarrow B = 1$

$u=-2 \Rightarrow \quad 4 = -4A \Rightarrow A = -1$

so $\quad I = \int \frac{1}{u-2} - \frac{1}{u+2} \, du$

$$I = \ln|u-2| - \ln|u+2| + C$$

$$= \ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C$$

2c $\int \sec^2 x \tan x \sqrt{1+\tan x}\ dx$ using the substitution $u^2 = 1+\tan x$

$\sqrt{1+\tan x} = u$

$\tan x = u^2 - 1$

since $\sec^2 x \equiv \tan^2 x + 1$

$\sec^2 x = \left\{ (u^2-1)^2 + 1 = u^4 - 2u^2 + 2 \right.$ but don't bother yet $\}$

$2u \frac{du}{dx} = \sec^2 x \Rightarrow$ "dx" is replaced by "$\frac{2u}{\sec^2 x} du$"

Hence $I = \int \sec^2 x \tan x \sqrt{1+\tan x}\ dx = \int \sec^2 x (u^2-1)(u) \frac{2u}{\sec^2 x} du$

And now you see why I said not to bother... the $\sec^2 x$ cancels

$\Rightarrow I = \int 2u^4 - 2u^2\ du = \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$

but since $u = \sqrt{1+\tan x}$ $\quad I = \frac{2}{5}(1+\tan x)^{5/2} - \frac{2}{3}(1+\tan x)^{3/2} + C$

$/ \square .$

2d $\quad I = \displaystyle\int \frac{\sqrt{x^2+4}}{x}\,dx$ using the substitution $u^2 = x^2 + 4$

differentiating $u^2 = x^2+4$ with respect to x gives

$$2u\frac{du}{dx} = 2x$$

$\Rightarrow$ we may replace "dx" with "$\dfrac{u}{x}du$"

Hence

$$I = \int \frac{\sqrt{x^2+4}}{x}\,dx = \int \frac{u}{x}\cdot\frac{u}{x}\,du$$

$$= \int \frac{u^2}{u^2-4}\,du$$

Now

$$\frac{u^2}{u^2-4} \equiv \begin{array}{r|l} & 1 \\ \cline{2-2} u^2+0u-4 & u^2+0u+0 \\ & -(u^2+0u-4) \\ \cline{2-2} & 4 \end{array}$$

$$\equiv 1 + \frac{4}{u^2-4} \equiv 1 + \frac{4}{(u+2)(u-2)}$$

And $\dfrac{4}{(u+2)(u-2)} \equiv \dfrac{A}{u+2} + \dfrac{B}{u-2}$

$\Rightarrow \quad 4 = A(u-2) + B(u+2)$

$u=2 \quad \Rightarrow \quad 4 = 4B \Rightarrow B=1$

$u=-2 \quad \Rightarrow \quad 4 = -4B \Rightarrow A=-1$

So

$$I = \int \frac{\sqrt{x^2+4}}{x}\,dx = \int \frac{u^2}{u^2-4}\,du$$

$$= \int 1 - \frac{1}{u+2} + \frac{1}{u-2}\,du$$

$$= u - \ln|u+2| + \ln|u-2| + C$$

$$= \sqrt{x^2+4} - \ln\left|\sqrt{x^2+4}+2\right| + \ln\left|\sqrt{x^2+4}-2\right| + C$$

$$\Rightarrow I = \sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + C$$

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2e $I = \int \sec^4 x \, dx$ using the substitution $u = \tan x$